LIGHTLIKE HYPERSURFACES IN SEMI RIEMANNIAN MANIFOLD WITH SEMI-SYMMETRIC NON METRIC CONNECTION

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ABSTRACT. In this paper, we study Lightlike hypersurfaces in semi Riemannian manifold admitting semi-symmetric non metric connection. We have shown that such manifolds are to be $\eta - Einstein$.

Keywords: Lightlike hypersurfaces, semi Riemannian manifold, semi-symmetric non metric connection.

1. INTRODUCTION

A complex manifold with constant holomorphic sectional curvature is called a complex space form and a contact metric manifold with constant $\phi$-sectional curvature is called a sasakian space form. In [1] Alegre et.al introduced and studied the generalised Sasakian space form as that almost contact metric manifold $\overline{M}(\phi, \xi, \eta, g)$ whose curvature tensor satisfies

$$\tilde{R}(X,Y)Z = \tilde{f}_1\{g(Y, Z)X - g(X, Z)Y\} + \tilde{f}_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X$$
$$+ 2g(X, \phi Y)\phi Z\} + \tilde{f}_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X$$
$$+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\},$$

for all vector fields $X, Y, Z$ and certain functions $f_1, f_2, f_3$ on $\overline{M}$. This generalises the notion of Sasakian space form as similar to generalised complex space form. The study of geometry of Lightlike hypersurfaces is interesting because of the fact the tangent and normal vector bundles intersect trivially. A linear connection non-zero torsion in a Riemannian manifold $\overline{M}$ is semi-symmetric if its torsion takes the form
where $\pi$ is a 1-form. If a semi Riemannian manifold $\overline{M}$ of index $k$ with $1 \leq k \leq n$ is semi-Riemannian metric $\overline{g}$ and semi-symmetric connection $\overline{\nabla}$ such that
\begin{equation}
(\overline{\nabla}_X)g(Y, Z) = -\pi(Y)g(X, Z) - \pi(X)g(Y, Z),
\end{equation}
then such a linear connection is called a non-metric connection given by [2]. We now suppose that the Semi-Riemannian manifold $\overline{M}$ admits a semi-symmetric non-metric connection given by
\begin{equation}
\overline{\nabla}_X Y = \nabla_X Y + \pi(Y)X
\end{equation}
for arbitrary vector fields $X, Y, Z$ on $M$, then the connection $\overline{\nabla}$ is called semi-symmetric non-metric connection on $\overline{M}$.

A complex manifold with constant holomorphic sectional curvature is known as a complex space form. An almost Hermitian manifold $M$ is called a generalized complex space form $M(f_1, f_2)$ if its Riemannian curvature tensor $R$ satisfies
\begin{equation}
R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, JZ)JY \\
- g(Y, JZ)JX + 2g(X, JY)JZ\}
\end{equation}
$\forall X, Y, Z \in TM$, where $f_1, f_2$ smooth functions on $M$.

Let $M$ be a hypersurface of a Kahler manifold $M^{2n}$.

A hypersurface of a Kahler manifold $M^{2n}$ is said to be Super quasi-umbilical if its second fundamental tensor has the form
\begin{equation}
H(X, Y) = \alpha g(X, Y) + \beta \omega(X)\omega(Y) + \gamma \delta(X)\delta(Y) + \rho \mathcal{D}(X, Y)
\end{equation}
where $\alpha, \beta, \gamma$ are scalars and the vector fields corresponding to 1-forms $\omega$ and $\delta$ are unit vector fields.

In this paper, we study generalised Sasakian space form which is a Lightlike hypersurface of complex manifold admitting semi-symmetric non metric connection.

2. LIGHTLIKE HYPERSURFACES OF COMPLEX SPACE FORMS

In a generalised Sasakian space form the following hold

$$\tilde{S}(Y, Z) = (n\tilde{f}_1 + 3\tilde{f}_2 - \tilde{f}_3)g(Y, Z) - (3\tilde{f}_2 + (n - 2)\tilde{f}_3)\eta(Y)\eta(Z),$$

$$\tilde{S}(\phi^2 Y, \phi^2 Z) = \tilde{S}(Y, Z) - \tilde{S}(\xi, Z)\eta(Y) - \tilde{S}(\xi, Y)\eta(Z) + \tilde{S}(\xi, \xi)\eta(Y)\eta(Z),$$

$$\tilde{S}(Y, \xi) = (n\tilde{f}_1 - (n - 1)\tilde{f}_3)\eta(Y),$$

$$\tilde{S}(\xi, \xi) = (n\tilde{f}_1 - (n - 1)\tilde{f}_3),$$

$$\tilde{R}(\xi, \phi Y, Z, \xi) = [\tilde{f}_1 - \tilde{f}_3]g(\phi Y, Z),$$

in this section we consider Lightlike hypersurface $M$ of an $n$-dimensional complex space form $\overline{M}$. Throughout this paper we assume that generalised Sasakian space form $M$ and we denote it by $M(f_1, f_2, f_3)$. Let $Q$ be the associated vector field of the 1-form $\pi$

$$\pi(X) = g(X, Q)$$

then we have the decomposition

$$\overline{Q} = \phi Q + \mu N,$$

where $Q$ and $\mu$ are respectively tangent vector field and $C^\infty$ function on $M$.

We assume that complex space form $\overline{M}$ admits a semi-symmetric non metric connection $\overline{\nabla}$. Let $\overline{\nabla}$ be the connection induced on $M(f_1, f_2, f_3)$. Then by Gauss formula

$$\overline{\nabla}_Y \phi Y = \phi(\nabla_X Y) + B(X, Y)N$$
for any $C^\infty$ vector fields $X$ and $Y$ on $M$, where $B$ is the second fundamental form on $M$ denoting by $\tilde{\nabla}$ the connection induced on the lightlike hypersurface from $\tilde{\nabla}$, we have the following Gauss formula with respect to the induced connection $\nabla$,

$$\tilde{\nabla}_{\phi X} Y = \phi(\nabla_X Y) + m(X, Y)N$$  \hspace{1cm} (2.8)

where $m$ is the tensor of the type $(0,2)$ of the lightlike hypersurface of $M$. In view of (1.4), we find

$$\tilde{\nabla}_{\phi X} Y = \nabla_{\phi X} Y + \tilde{\pi}(\phi Y)\phi X.$$  

Using (2.7) and (2.8), we can also write

$$\phi(\nabla_X Y) + m(X, Y)N = \phi(\nabla_X Y) + B(X, Y)N + \tilde{\pi}(\phi Y)\phi X.$$  \hspace{1cm} (2.9)

Substituting (2.6) into (2.9), we obtain

$$\phi(\nabla_X Y) + m(X, Y)N = \phi(\nabla_X Y + \pi(Y)X) + B(X, Y)N.$$  

From which we get

$$\tilde{\nabla}_X Y = \nabla_X Y + \tilde{\pi}(Y)X,$$  \hspace{1cm} (2.10)

$$\tilde{\pi}(X) = \tilde{\pi}(\phi X)$$

$$m(X, Y) = B(X, Y).$$  \hspace{1cm} (2.11)

Taking account of the fact that connection induced on lightlike hypersurface from Levi-Civita connection is not metric, we obtain from (2.10) the following

$$(\nabla_X g)(Y, Z) = m(X, Y)\eta(Z) + m(X, Z)\eta(Y) - \pi(Y)g(X, Z) - \pi(Z)g(X, Y),$$  \hspace{1cm} (2.12)

where $\eta(Z) = \tilde{g}(Z, N)$ for any $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(trTM^{-1}).$

We also get from (2.10),

$$T(X, Y) = \tilde{\pi}(Y)X - \tilde{\pi}(X)Y$$  \hspace{1cm} (2.13)
Also the Weingarten formula [6] with respect to the semi-symmetric non-metric connection is given by

\[ \tilde{\nabla}_{\phi X}N = -\phi(A_N X) + \tau(X)N. \]  

(2.14)

By definition of the curvature tensor \( \tilde{R} \) on \( M \), we have

\[ \tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z. \]

Putting \( X = \phi X, Y = \phi Y, Z = \phi Z \), we get

\[ \tilde{R}(\phi X, \phi Y)\phi Z = \tilde{\nabla}_{\phi X} \tilde{\nabla}_{\phi Y} \phi Z - \tilde{\nabla}_{\phi Y} \tilde{\nabla}_{\phi X} \phi Z - \tilde{\nabla}_{\phi [X,Y]}\phi Z \]

Thus using (2.8) and (2.14), we find

\[ \tilde{R}(\phi X, \phi Y)\phi Z = \phi(\tilde{R}(X, Y)Z) + m(X, Z)A_N Y - m(Y, Z)A_N X \]

\[ + \{m(\pi(Y)X - \pi(X)Y, Z) + (\nabla_X m)(Y, Z) \]

\[ - (\nabla_Y m)(X, Z) + m(Y, Z)\tau(X) - m(X, Z)\tau(Y)\}N, \]

(2.15)

is the curvature tensor of lightlike hypersurface with a semi-symmetric non-metric connection \( \tilde{\nabla} \).

Contracting the above with respect to \( W \)

\[ -\tilde{R} (\phi X, \phi Y, \phi Z, W) = -g(\phi(R(X, Y)Z, W)) + m(X, Z)g(A_N X, W) - m(Y, Z)g(A_N W, W). \]

(2.16)

Using \( g(A_N Y, W) = g(m(Y, W), N) \) and \( m(X, Y) = H(X, Y)N \),

where \( H \) is second fundamental form on \( M \)

(2.16) reduces to

\[ -\tilde{R} (\phi X, \phi Y, W, \phi Z) = -R(X, Y, Z, \phi W) + m(X, Z)H(Y, W) - m(Y, Z)H(X, W). \]  

(2.17)
Using (1.5) and (1.6) in (2.17), we get
\[-\tilde{R}(\phi X, \phi Y, W, \phi Z) = - [f_1 \{g(Y, Z)g(X, \phi W) - g(X, Z)g(Y, \phi W) \\
+ f_2 \{g(X, JZ)g(JY, \phi W) - g(Y, JZ)g(JX, \phi W) \\
+ 2g(X, JY)g(JZ, \phi W)\}] + m(X, Z)[\alpha g(Y, W) \\
+ \beta \omega(Y)\omega(W)] - m(Y, Z)[\alpha g(X, W) + \beta \omega(X)\omega(W)].\] (2.18)

Putting \(X = Z = e_i, i = 1, ..., n\), we obtain
\[-\tilde{R}(\phi e_i, \phi Y, W, \phi e_i) = - [f_1 (1 - n)g(Y, \phi W) - 3f_2 g(JY, J\phi W)] \\
+ \text{Tra} \ m[\alpha g(Y, W) + \beta \omega(Y)\omega(W)] \\
- \alpha m(Y, W) - \beta m(Y, \rho_1)\omega(W),\] (2.19)

where \(\rho_1\) is corresponding characteristic vector field of \(\omega\), i.e \(\omega(X) = g(X, \rho_1)\).

Now
\[-\tilde{R}(\phi e_i, \phi Y, W, \phi e_i) = \sum_{i=1}^{n+1} \tilde{R}(\phi e_i, \phi Y, W, e_i) - \tilde{R}(\xi, \phi Y, W, \xi) \\
- \tilde{R}(\phi e_i, \phi Y, W, \phi e_i) = \tilde{S}(\phi Y, W) - \tilde{R}(\xi, \phi Y, W, \xi)\] (2.20)

Using (2.20) and (2.5) in (2.19)
\[\tilde{S}(\phi Y, W) = (\tilde{f}_1 - \tilde{f}_3 - f_1 (1 - n))g(\phi Y, W) - 3f_2 g(JY, J\phi W) - \text{Tra} \ m[\alpha g(Y, W) \\
+ \beta \omega(Y)\omega(W)] + \alpha m(Y, W) - \beta m(Y, \rho_1)\omega(W).\] (2.21)

If \(\phi X\) is orthogonal \(\xi\) then \(\omega(\phi X) = 0\), Therefore replacing \(W\) by \(\phi W\) in (2.21)
\[\tilde{S}(\phi Y, \phi W) = (\tilde{f}_1 - \tilde{f}_3 - f_1 (1 - n) + 3f_2)(g(\phi Y, W) - \eta(Y)\eta(W)) \\
- \text{Tra} \ m(\alpha g(Y, \phi W)) + \alpha m(Y, \phi W).\] (2.22)

Replace \(Y\) by \(\phi Y\) and \(W\) by \(\phi W\) in (2.22), we obtain
\[\tilde{S}(\phi^2 Y, \phi^2 W) = (\tilde{f}_1 - \tilde{f}_3 - f_1 (1 - n) + 3f_2)(g(\phi Y, \phi W)) \\
- \text{Tra} \ m(\alpha g(\phi Y, \phi^2 W)) + \alpha m(\phi Y, \phi^2 W).\] (2.23)
Using (2.1), (2.2), (2.3), (2.4), we get

\[
\tilde{S}(Y, W) = (\tilde{f}_1 - \tilde{f}_3 - f_1(1 - n) + 3f_2)(g(Y, W)) \\
+ (\tilde{f}_1(n - 1) + \tilde{f}_3(2 - n) + f_1(1 - n) - 3f_2)\eta(Y)\eta(W) \\
-Tra \ m(\alpha g(\phi Y, W)) + \alpha m(\phi Y, W).
\] (2.24)

Put (2.1) in (2.24), we obtain

\[
\alpha m(\phi Y, W) = [(f_1 - \tilde{f}_1)(n - 1) + 3(f_2 - \tilde{f}_2)](g(Y, W)) \\
-[(f_1 - \tilde{f}_1)(n - 1) + 3(f_2 - \tilde{f}_2)]\eta(Y)\eta(W) \\
-Tra \ m(\alpha g(\phi Y, W)).
\] (2.25)

Put (2.25) in (2.24) to get

\[
\tilde{S}(Y, W) = (n\tilde{f}_1 + 3\tilde{f}_2 - \tilde{f}_3)(g(Y, W)) + (\tilde{f}_3(2 - n) - 3\tilde{f}_2)\eta(Y)\eta(W) \\
+ Tra \ m(\alpha g(\phi Y, W)).
\] (2.26)

Put (2.1) in (2.26), We obtain

\[
Tra \ m(\alpha g(\phi Y, W)) = (6\tilde{f}_2 - 2(2 - n)\tilde{f}_3)\eta(Y)\eta(W).
\] (2.27)

We use (2.27) in (2.26) to obtain

\[
\tilde{S}(Y, W) = [n\tilde{f}_1 + 3\tilde{f}_2 - \tilde{f}_3](g(Y, W)) + [3\tilde{f}_2 - \tilde{f}_3(2 - n)]\eta(Y)\eta(W).
\]

**Theorem 2.1.** Let $M$ be a lightlike hypersurface of a complex space form $\bar{M}$. If $M$ is a generalised Sasakian space form then it is $\eta$-Einstein.
3. CURVATURE TENSOR ADMITTING SEMI-SYMMETRIC NON METRIC CONNECTION

The curvature tensor in generalised sasakian space form admitting semi-symmetric non-metric connection is given by

\[
\tilde{R}(X, Y, Z, W) = f_1\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
+ f_2\{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) + 2g(X, \phi Y)g(\phi Z, W)\} \\
+ f_3\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + \eta(Y)\eta(W)g(X, Z) \\
- \eta(X)\eta(W)g(Y, Z)\} + s(X, Z)g(Y, W) - s(Y, Z)g(X, W),
\]

(3.1)

for any vector fields \(X, Y, Z, W\), where \((0,2)\)-tensor field \(S\) is given by

\[
S(X, Y) = (\nabla_X \phi)Y - \phi(X)\phi(Y).
\]

(3.2)

Suppose \(\tilde{M}\) is a generalised complex space form admitting semi-symmetric non-metric connection \(\tilde{\nabla}\). Then the curvature tensor in \(M\) is given by

\[
\tilde{R}(X, Y, Z, W) = F_1\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
+ F_2\{g(X, JZ)g(JY, W) - g(Y, JZ)g(JX, W) + 2g(X, JY)g(JZ, W)\} \\
+ s(X, Z)g(Y, W) - s(Y, Z)g(X, W).
\]

(3.3)

From (2.17) in (3.3), we get

\[
-\tilde{R}(\phi X, \phi Y, W, \phi Z) = - [F_1\{g(Y, Z)g(X, \phi W) - g(X, Z)g(Y, \phi W)\} \\
+ F_2\{g(X, JZ)g(JY, \phi W) - g(Y, JZ)g(JX, \phi W) + 2g(X, JY)g(JZ, \phi W)\} \\
+ s(X, Z)g(Y, \phi W) - s(Y, Z)g(X, \phi W)] \\
+ m(X, Z)H(Y, \phi W) - m(Y, Z)H(X, \phi W).
\]
Taking $X = Z = e_i$ in the above equation and using (1.6), we obtain
\[
-\tilde{R}(\phi e_i, \phi Y, W, \phi e_i) = -[F_1(1 - n)g(Y, \phi W) - 3F_2 g(JY, J\phi W) + Tra \ s g(Y, \phi W) - s(Y, \phi W)]
\]
\[+trace \ m[\alpha g(Y, W) + \beta \omega(Y)\omega(W)] - \alpha m(Y, W) - \beta m(Y, \rho_1)\omega(W),
\]
where $\rho_1$ is corresponding characteristic vector field and $\omega(X) = g(X, \rho_1)$

Now
\[
\tilde{R}(\phi e_i, \phi Y, W, \phi e_i) = \sum_{i=1}^{n+1} \tilde{R}(\phi e_i, \phi Y, W, \phi e_i) = \tilde{S}(\phi Y, W) - \tilde{R}(\xi, \phi Y, W, \xi).
\]

Using (3.4) and (3.5), we obtain
\[
\tilde{S}(\phi Y, W) = -(f_1 - f_3) + F_1(1 - n)g(Y, \phi W) - 3F_2 g(JY, J\phi W) + Tra \ s g(Y, \phi W)
\]
\[-\alpha m(\alpha g(Y, W) + \beta \omega(Y)\omega(W)] + \alpha m(Y, W) - \beta m(Y, \rho_1)\omega(W).
\]

If $\phi X$ is orthogonal $\xi$ then $\omega(\phi X) = 0$. Therefore replacing $W$ by $\phi W$, we get
\[
\tilde{S}(\phi Y, \phi W) = -(f_1 - f_3) + F_1(1 - n) - 3F_2 + Tra \ s(g(Y, W) - \eta(Y)\eta(W))
\]
\[-\alpha m(\alpha g(Y, \phi W)) + \alpha m(Y, \phi W)
\]

Replace $Y$ by $\phi Y$ and $W$ by $\phi W$ to get
\[
\tilde{S}(\phi^2 Y, \phi^2 W) = -(f_1 - f_3) + F_1(1 - n) - 3F_2 + Tra \ s\eta(Y)\eta(W)
\]
\[-\alpha m(\alpha g(\phi Y, \phi^2 W)) + \alpha m(\phi Y, \phi^2 W).
\]

Using (3.6), we get
\[
\tilde{S}(Y, W) = (f_1 - f_3)(\eta(Y)\eta(W)) + n[\eta(Y)W + \eta(W)Y - 2\eta(Y)\eta(W)\xi]
\]
\[+(-f_1 + f_3) + F_1(1 - n) - 3F_2 + Tra \ s(\eta(Y)\eta(W) - g(Y, W))
\]
\[-\alpha m(\alpha g(Y, \phi W)) - \alpha m(\phi Y, W).
\]
Theorem 3.1. Let $M$ be a lightlike hypersurface of complex space form $\bar{M}$ admitting semi-symmetric non-metric connection. If $M$ is a generalised Sasakian space form then its Ricci tensor is given by (3.6). Further for $\alpha = 0$, $M$ reduces to $\eta$-Einstein.

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